Problem Set 3 --- Suggested Answers

1. In chapters 15 – 18 the treatment is generalized to unbounded production technologies, resulting in the observation that when general equilibrium prices are announced, firms discover that profit maximization is consistent with finite production plans, even though the firm's technology admits unbounded production. Prices communicate scarcity, enough so that they are sufficient to communicate this message.

As pointed out in class discussion, this argument could have been undertaken in a different order: Start with chapter 15, demonstrating that the attainable production plans are bounded. Then consider $\tilde{S}^j(p)$ as determined by $\tilde{Y}^j$. Then the argument of Chapters 11 to 14 can be applied directly. Return then to Chapters 15 – 18, establishing the equivalence at general equilibrium prices p* of $D^i(p^*)$ to $\tilde{D}^i(p^*)$ and of $S^j(p^*)$ to $\tilde{S}^j(p^*)$.

Suggested Answers: 18.19, 18.20, 18.21

18.19 In Chapters 14 and 18 we used the mapping $T : P \to P$ as a price adjustment function whose fixed points are competitive equilibria. Consider instead using the mapping $\Gamma : P \to P$ where the ith co-ordinate mapping of Γ is $\Gamma_i(p) = \frac{med\left[0, p_i + p_i \widetilde{Z}_i(p), c\right]}{\sum_{j=1}^{N} med\left[0, p_j + p_j \widetilde{Z}_i(p), c\right]}$ where 'med' stands for 'median' (the middle value of the three in brackets; when two of the three are equal, that value is the median) and c is defined as in Chapter 12 as a strict upper bound on the Euclidean length of an attainable consumption. Assume $c > 1$. Assume that Walras' Law holds as an equality, that is, that $p \cdot \widetilde{Z}(p) = 0$.

1. Show that every competitive equilibrium price vector p^0 is a fixed point of Γ.
   Suggested Answer: $\tilde{Z}_i(p^0) \leq 0$ so $\Gamma_i(p^0) = \frac{med\left[0, p_i^0 + p_i^0 \widetilde{Z}_i(p^0), c\right]}{\sum_{j=1}^{N} med\left[0, p_j^0 + p_j^0 \widetilde{Z}_i(p^0), c\right]} =$ p_i^0 if $p_i^0 > 0$ and $= 0$ when $\tilde{Z}_i(p^0) < 0$. Thus p^0 is a fixed point.
2. A vertex of the price simplex is a co-ordinate unit vector, a vector of the form (0,0,...,0,1,0,...,0), with 1 in one co-ordinate and 0 in all others. Show that every vertex of the price simplex P is a fixed point of Γ.
   Suggested Answer: Denote the i^{th} unit vector e_i.
   $\Gamma_i(e_i) = \frac{med\left[0, 1 + \widetilde{Z}_i(e_i), c\right]}{\sum_{j=1, j \neq i}^{N} med[0,0,c] + med[0, 1 + \widetilde{Z}_i(e_i), c]} = \frac{1 + \widetilde{Z}_i(e_i)}{1 + \widetilde{Z}_i(e_i)} = 1$.And for $j \neq i$, $\Gamma_i(e_i) = 0$ since 0 is the value of the numerator. Hence e_i is a fixed point of Γ.
3. Under the usual assumptions of continuity of $\tilde{Z}(p)$, $\Gamma(\cdot)$ can be shown to have a fixed point, $p^* = \Gamma(p^*)$. Does this prove that the economy — under those sufficient conditions — has a competitive equilibrium?
   Suggested Answer: Parts 1 and 2 have demonstrated that both general equilibrium prices and co-ordinate unit vectors are fixed points of Γ. There is no reason to believe that co-ordinate unit vectors are general equilibria. Hence the existence of a fixed point of Γ is uninformative about the existence of general equilibrium.

18.20 Consider a tax and public good provision program. Using the model of chapters 15 – 18, let each household $i \in H$ be taxed, in kind, $0.1r^i$, so that household income is $M^i(p) = p \cdot (.9r^i) + \sum_{j \in F} \alpha^{ij} \pi^j(p)$. The resources $.1 \sum_{i \in H} r^i$ are then used to provide a public good, γ, according to the production function $\gamma = g(.1 \sum_{i \in H} r^i)$. We take g to be continuous, concave.

Household utility functions are then characterized as $u^i(x^i; \gamma)$. The households treat γ parametrically. Assume all the usual properties of u^i, particularly continuity in its arguments. The household budget constraint is then $p \cdot x^i \leq M^i(p)$.

1. Define a competitive equilibrium with public goods for this economy.
   Suggested Answer:

$$\{p^\circ, x^{\circ i}, y^{\circ j}\}, p^\circ \in \mathbf{R}^N_+, i \in H, j \in F,$$

is said to be a competitive equilibrium if

(i) $y^{\circ j} \in Y^j$ and $p^\circ \cdot y^{\circ j} \geq p^\circ \cdot y$ for all $y \in Y^j$, for all $j \in F$,

(ii) $\gamma = g(.1 \sum_{i \in H} r^i)$

(iii) $x^{\circ i} \in X^i$, $p^\circ \cdot x^{\circ i} \leq M^i(p^\circ) = (0.9) \times p^\circ \cdot r^i + \sum_{j \in F} \alpha^{ij} p^\circ \cdot y^{\circ j}$ and $u^i(x^{\circ i}, \gamma) \geq u^i(x, \gamma)$ for all $x \in X^i$ with $p^\circ \cdot x \leq M^i(p^\circ)$ for all $i \in H$, and

(iv) $0 \geq \sum_{i \in H} x^{\circ i} - \sum_{j \in F} y^{\circ j} - \sum_{i \in H} (0.9) \times r^i$ with $p^\circ_k = 0$ for coordinates k so that the strict inequality holds.

2. Assuming the usual properties on production and consumption, does Theorem 18.1, Existence of Equilibrium, still hold? Explain.
   Suggested Answer: Assume that the assumptions of Theorem 18.1 are fulfilled for the economy without taxation and public goods. The only conditions changed by the addition of lump-sum taxation and public goods is adequacy of income, C.VII, and convexity, insatiability and continuity of preferences. Check to be sure that C.VII is fulfilled after taxation. Then assume that $u^i(x^i, \gamma)$ is continuous, concave, and insatiable for all values of γ in the relevant range. If these augmented conditions are fulfilled, then the assumptions of Theorem 18.1 are fulfilled and there exists a competitive general equilibrium.

18.21 Consider an economy with a finite number of households (enough so that it makes sense for them to be price-takers), two firms acting as price-takers, and two outputs, X and Y. Each household is endowed with one unit of labor, which it sells on a competitive labor market. The household then uses its budget to buy X and Y. All households have the same Cobb-Douglas utility function $U^i(X^i, Y^i) = X^i Y^i$.

X is produced using the technology $X = [L^x]^2$, where $L^x \geq 0$ is the labor used as an input to X production and the superscript "2" indicates a squared term.

Y is produced using the technology $Y = [L^y]^2$, where $L^y \geq 0$ is the labor used as an input to Y production and the superscript "2" indicates a squared term.

Note that each of these technologies displays scale economy.
There is no competitive equilibrium in this example. Why? Is this a counterexample to Theorem 18.1? If not, which assumptions of Theorem 18.1 are not fulfilled? Explain.
Suggested Answer: No this is not a counterexample. The assumptions of Theorem 18.1 include convexity of technology. The scale economies in this example generate a non-convex technology.

19.1. (starting from Problem 14.2) The redistributive income taxation (with the present model's definition of income as the market value of endowment) represents merely a redistribution of endowment across households, followed by implementation of a competitive equilibrium allocation. The First Fundamental Theorem of Welfare Economics applies and the allocation is Pareto efficient.

(starting from Problem 14.3) The first fundamental theorem of welfare economics is inapplicable because τ represents a wedge between buying and selling prices. The essential step in the proof of the 1FTWE is the notion that if an alternative attainable allocation were more expensive at the household level than the equilibrium allocation, *then it would necessarily be more profitable to produce (or sell from endowment) than the equilibrium allocation.* That inference breaks down in the presence of excise taxes since the tax represents a wedge between buying and selling prices. The equilibrium need not be Pareto efficient.

19.13 It is no longer true that $p^o \cdot w^{oi} = p^o \cdot r^i + \sum_{j \in F} \alpha^{ij}(p^o \cdot y^{oj})$. Thus the notion that a preferable, more expensive, production plan must be more profitable fails.